Elementary Mathematics:
Properties of Number Ranges

Proof of the Irrationality of the Square Root of 2 ($\sqrt{2} \in \mathbb{R}$)

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For the proof of the irrationality of the square root of 2 different proofs exist. This article explains the indirect proof variant in detail and introduces one of the possible short spellings in mathematical notation. Exercises with solutions consolidate what has been learned.

Contents

1 The Indirect Proof

1.1 The Procedure of the Proof of Contradiction

The “law of the excluded middle” (Tertium non datur) states that a statement and its negation cannot both be false and that one of them must be true.

In indirect reasoning, it is assumed that the assertion to be proved is false. Then this assumption leads to a contradiction.

But if the negation of a statement leads to a contradiction, then the statement must be true. According to the *law of the excluded middle*, there can be no third possibility.

Beside binary logic also many-valued logics resp. n-valued logics[1] exist.

1.2 On the Notation of Proofs

Proof exercises usually begin with “Show that” and “Proove that” or with “Proove:“. In German texts you will also find “Z. z.:”, the abbreviated form of “Zu zeigen:” (“To show:”).

In the case of indirect proofs, the contradiction that arises is marked with a thunderbolt: $\lightning$.

The conclusion of a proof is marked either with the abbreviation “q.e.d”, which stands for the Latin words *quod erat demonstrandum*, in English: “what was to be shown”. Or a white or black square is used as the final symbol: $\square$.

1.3 Definition W. l. o. g.

W. l. o. g., “without loss of generality” is a hint used in mathematical proofs which states that the specific case considered in the proof also covers all other possible cases

Thus one calculates exemplarily a case, which at the same time stands for all other possibilities. The case dealt with in the proof is therefore characteristic and valid for all other cases without exception.

2 Proof of the Irrationality of the Square Root of 2

2.1 Introduction

We want to show, that $\sqrt{2}$ cannot be represented as a fraction, that $\sqrt{2} \notin \mathbb{Q}$.

Fractions have the form $\frac{a}{b}$ and originate from the set of rational numbers, $\mathbb{Q}$.

[1] Fundamental work: Gotthard Günther, “Idee und Grundriß einer nicht-Aristotelischen Logik. 1. Band: Die Idee und ihre philosophischen Voraussetzungen.”, Verlag von Felix Meiner, Hamburg 1959.

In the numerator is a whole number.[2] The denominator b must be a natural number,[3] to exclude division by zero:

$$\mathbb{Q} := \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \right\}$$

Read: "Q is the set of all elements (of all fractions) a over b for which a (is) element (of) Z and b (is) element (of) N".

We now show that $\sqrt{2}$ does not belong to the set of rational numbers, but to the superset of $\mathbb{Q}$, the set of real numbers, $\mathbb{R}$.

2.2 Proof of the Irrationality of $\sqrt{2}$

Show: $\sqrt{2}$ is irrational.

Proof: indirect. We assume that $\sqrt{2}$ is *not* irrational.

1. Then $\sqrt{2}$ is rational ($\in \mathbb{Q}$) and can be written as fraction:

$$\sqrt{2} = \frac{a}{b}$$

2. Without loss of generality[4] we assume that $\frac{a}{b}$ is a shortened fraction, because a shortened fraction exists for every fraction. This means that a and b are relatively prime, that

[2] The set of integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

[3] The set of natural numbers without zero: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

[4] A general definition of w. l. o. g. is given in section 1.3 on the facing page. Explanation of w. l. o. g. in the frame of reference of this proof: The assumption that the fraction $\frac{a}{b}$ can be reduced to prime factors also applies to all other possible cases, i.e. to all other roots of *non-square numbers*, such as $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, etc. One reads the following explanation, why only roots from non-square numbers are allowed, only after the internalization of the proof in the main text. If we would also allow quadratic numbers, then the assumption that the root of the quadratic number can be represented as a fraction of two coprime numbers would not lead to a contradiction. Example: The 4 is a square number, $4 = 2 \cdot 2$, and can be represented as coprime fraction, for example $\sqrt{4} = \frac{8}{4} = \frac{2}{1} = 2$.

$$\begin{aligned} \sqrt{4} &= \frac{a}{b} && \mid (\ )^2 \text{ squaring} \\ 4 &= \frac{a^2}{b^2} && \mid \cdot b^2 \\ 4b^2 &= a^2 \end{aligned}$$

From the equation $4b^2 = a^2$ it follows, that a^2 is an even number $(2 \cdot 2 \cdot b^2 = a^2)$, and therefore also the numerator a must be even. Therefore we set $a = 2k$, resolve to b^2 and find that that b is an odd number and therefore *no* contradiction arises, because numerator and denominator remain relatively prime:

$$\begin{aligned} 4b^2 &= (2k)^2 \\ 4b^2 &= 4k^2 && \mid \div 4 \\ b^2 &= k^2 && \mid \sqrt{\ } \\ b &= k \end{aligned}$$

they have no common denominator other than 1.[5]

3. We now transform the equation to a^2:

$$\begin{aligned} \sqrt{2} &= \tfrac{a}{b} && \mid (\ )^2 \text{ squaring} \\ 2 &= \left(\tfrac{a}{b}\right)^2 \\ 2 &= \tfrac{a^2}{b^2} && \mid \cdot b^2 \\ 2b^2 &= a^2 \end{aligned}$$

From the equation $2b^2 = a^2$ it follows that a^2 must be an even number. An even number namely can be defined as $\pm 2k$, with $k \in \mathbb{N}_0$.[6]

The 2 is thus a divisor of a^2, contained in the divisor set of a^2.[7]

4. We now use the following elementary facts for our further conclusion: If a number is even, then its square is even.[8] If a number is odd, its square is also odd.[9]

Since a^2 is an even number, it follows that the numerator variable a^2 must also be an even number.[10]

5. Since the 2 is a divisor of a, so it is an even number, we can replace the a in $2b^2 = a^2$

[5] Examples of reduced coprime fractions: $\frac{6}{10} = \frac{3}{5}$, the greatest common divisor of 3 and 5 is the 1: $\gcd(3{,}5) = 1$. $\frac{30}{5} = \frac{6}{1}$, $\gcd(6{,}1) = 1$. For $\frac{17}{13}$ holds: $\gcd(17{,}13) = 1$. The cases with a 0 in the numerator $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots\}$ are taken into account, the zero is divisible by all numbers: $\forall x \in \mathbb{N}$ (generally $\forall x \in \mathbb{Z}, \mathbb{R}, \mathbb{C}$) holds true "$x$ divides 0". Fractions of form $\frac{0}{x}$ (with $x \in \mathbb{N}$) can no longer be shortened. Since $1 \mid 0$ ("1 divides 0") applies in the denominator and $1 \mid x$ in the denominator, $\frac{0}{x}$ is always coprime, i.e. it has only the 1 as greatest common divisor: $\gcd(0, x) = 1$.

[6] Even numbers are multiples of two, they have the form $2 \cdot k$. The set of possible natural numbers for k also includes the zero ($k \in \mathbb{N}_0$), so that besides positive and negative even numbers also the case $2 \cdot 0 = 0$ is possible. The zero is therefore also considered to be an even number. For our b^2, of course, $b^2 \geqq 1$ still applies, since $b > 0$ must apply ($b \in \mathbb{N}$), so that no division by zero can occur.

[7] $2 \mid a^2$ resp. $2 \in \mathrm{T}_{a^2}$.

[8] Examples for $x = (2k)^2$, with $k \in \mathbb{N}$: $x = 2, x^2 = 4$; $x = 4, x^2 = 16$; $x = 6, x^2 = 36$.

[9] Examples for $x = (2k+1)^2$, with $k \in \mathbb{N}_0$: $x = 1, x^2 = 1$; $x = 3, x^2 = 9$; $x = 5, x^2 = 25$.

[10] For even and odd numbers, each prime factor ($p \geqq 2$) of a in a^2 must occur at least twice. Even numbers also always contain 2 as prime factor. Examples for even numbers:
$a = 2\,(= 2), a^2 = 4\,(= 2 \cdot 2)$;
$a = 4\,(= 2 \cdot 2), a^2 = 16\,(= 2 \cdot 2 \cdot 2 \cdot 2)$;
$a = 6\,(= 2 \cdot 3), a^2 = 36\,(= 2 \cdot 2 \cdot 3 \cdot 3)$;
$a = 8\,(= 2 \cdot 2 \cdot 2), a^2 = 64\,(= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2)$;
$a = 10\,(= 2 \cdot 5), a^2 = 100\,(= 2 \cdot 2 \cdot 5 \cdot 5)$
$a = 30\,(= 2 \cdot 3 \cdot 5), a^2 = 900\,(= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5)$.
Examples of odd numbers:
$a = 3\,(= 3), a^2 = 9\,(= 3 \cdot 3)$;
$a = 5\,(= 5), a^2 = 25\,(= 5 \cdot 5)$;
$a = 15\,(= 3 \cdot 5), a^2 = 225\,(= 3 \cdot 3 \cdot 5 \cdot 5)$.
The respective prime factors of a square number a^2 therefore also occur in the base number a.

with $2k$ ($k \in \mathbb{N}_0$):

$$\begin{aligned} 2b^2 &= a^2 \\ 2b^2 &= (2k)^2 \\ 2b^2 &= 4k^2 \qquad | \div 2 \\ b^2 &= 2k^2 \ \lightning \end{aligned}$$

The contradiction has now occurred in the last line: If $b^2 = 2k^2$ is valid, then b^2 and therefore the denominator variable b are even as well. In our assumption, however, we have assumed that our fraction $\frac{a}{b}$ is already coprime, i.e. that it exists in reduced form. But if both the numerator *and* the denominator are even, then the fraction can still be shortened.[11]

Thus the assumption that $\sqrt{2}$ is not irrational, is therefore wrong. According to the *law of the excluded middle* we have thereby proven, that the first statement is true: $\sqrt{2}$ is irrational.

$\square$

2.3 Proof in Short Notation

When writing mathematical texts, it is advantageous to note the contents in parallel in pure mathematical notation. This makes the document accessible to an international readership.

Quantity notations can vary, in addition irrational numbers are sometimes summarized in their own set denoted by a large $\mathbb{I}$: $\mathbb{I} = \{\sqrt{2}, \pi, \dots\}$.[12]

2.3.1 Explanation of the Mathematical Symbols

The sign for divisibility, $|$. Example: $2 \mid 4$, read "2 divides 4". $2 \nmid 5$, read "2 does not divide 5". In general: $a \mid b$, "a divides b". $a \nmid b$, "a does not divide b".

Coprimeness, $\perp$. Example: $3 \perp 5$, read "3 is relatively prime to 5" resp. "3 is coprime to 5" resp.. "3 and 5 are mutually prime". $3 \not\perp 6$, read "3 is not relatively prime to 6" resp. "3 is not coprime to 6" resp. "3 and 6 are not mutually prime". Generally: $a \perp b$, "a is relatively prime to b" resp. "a is coprime to b" resp. "a and b are mutually prime". $a \not\perp b$, read "a is not relatively prime to b" resp. "a is not coprime to b" resp. "a and b are not mutually prime".[13]

[11] Examples of unreduced even fractions $\frac{a}{b}$, with $a \in \mathbb{Z}$ und $b \in \mathbb{N}$: $\{\dots, \frac{-4}{4}, \frac{-2}{4}, \dots, \frac{-4}{2}, \frac{-2}{2}, \frac{2}{2}, \frac{4}{2}, \dots, \frac{2}{4}, \frac{4}{4}, \dots\}$.

[12] In this article, the sans serif variant of "dsfont" is mainly used for number sets, following the double dash notation for blackboard notation: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. The font (the character set) "amsb" is widespread: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. In older literature, fracture letters are mostly used as symbols for sets of numbers.

[13] In geometry, the symbol $\perp$ indicates orthogonality (perpendicularity) between vectors. Example: $\vec{a} \perp \vec{b}$, "$\vec{a}$ is orthogonal to $\vec{b}$".

The universal quantifier $\forall$ originates from the propositional logic and is read as "for all". Example: $\forall x \in A$: $2 \mid x$.

The (sign of) implication, $\Rightarrow$. Example: $A \Rightarrow B$, read "if A then B" resp. "A implies B".

The logical equivalence, $\Leftrightarrow$. Example: $A \Leftrightarrow B$, read "A is equivalent to B" resp. "A and B are equivalent."

W. l. o. g., "without loss of generality": See section 1.3 on page 2 and footnote 4.

Partly there are additional ways to read expressions.

2.3.2 Proof $\sqrt{2} \in \mathbb{IR}$, in abbreviated Form

It follows one of several possible transcripts of the proof in abbreviated form.

Show: $\sqrt{2}$ is irrational ($\sqrt{2} \in \mathbb{IR}$).

Proof: indirect. We assume that $\sqrt{2}$ is not irrational:

1. Assumption: $\sqrt{2}$ is not irrational $\Rightarrow \sqrt{2} \in \mathbb{Q}$, i.e. $\sqrt{2} = \frac{a}{b}$.
2. Wlog: $a \perp b$.
3. $\sqrt{2} = \frac{a}{b} \Leftrightarrow 2b^2 = a^2$.
4. $(2b^2 = a^2) \Rightarrow 2 \mid a^2$, as $a^2 = \pm 2k$, with $k \in \mathbb{N}_0$.
5. $\forall x$ with $x = 2k$, $k \in \mathbb{N}_0 : x^n = 2k$ rsp. $2 \mid x^n$.
6. $\Rightarrow 2 \mid a$, and: $2b^2 = a^2 \Leftrightarrow 2b^2 = (2k)^2$.
7. $\Rightarrow b^2 = 2k^2 \Rightarrow a \not\perp b \Rightarrow \lightning$ to 2.

□

For a more detailed illustration, the individual calculation steps can also be written above the equivalence arrows.

Show: $\sqrt{13} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt{13} \in \mathbb{Q}$. $\Rightarrow \sqrt{13} = \frac{a}{b}$.
2. Wlog: $a \perp b$.
3. $\sqrt{13} = \frac{a}{b} \overset{()^2}{\Leftrightarrow} 13 = \frac{a^2}{b^2} \overset{\cdot b^2}{\Leftrightarrow} 13b^2 = a^2 \overset{\div a}{\Leftrightarrow} 13\frac{b^2}{a} = a \Rightarrow 13 \mid a$.
4. $13b^2 = a^2 \overset{a:=13k}{\Longleftrightarrow} 13b^2 = (13k)^2 \Leftrightarrow 13b^2 = 169k^2 \overset{\div 13}{\Leftrightarrow} b^2 = 13k^2 \overset{\div b}{\Leftrightarrow} b = 13 \cdot \frac{k^2}{b} \Rightarrow 13 \mid b$.
5. $\Rightarrow a \not\perp b \Rightarrow \lightning$ to 2.

□

3 Exercises with Solutions

Of the following exercises, the first four were taken from [1].[14]

1. Show: $\sqrt{3}$ is irrational.

Proof: indirect.

1. Assumption: $\sqrt{3} \in \mathbb{Q}$. Then follows: $\sqrt{3} = \frac{a}{b}$ with $a \in \mathbb{Q}$, $b \in \mathbb{N}$; a and b are reduced and coprime.

2.
$$\begin{aligned} \sqrt{3} &= \frac{a}{b} \quad | \; (\;)^2 \text{ squaring} \\ 3 &= \frac{a^2}{b^2} \quad | \cdot b^2 \\ 3b^2 &= a^2 \end{aligned} \tag{1}$$

3. Isolating the numerator variable: From $3b^2 = a^2$ follows $3 \mid a^2$, as the 3 in $3 \cdot b^2 = a^2$ is divisor (prime factor) of a.

4. From $3 \mid a^2$ follows $3 \mid a$:

$$\begin{aligned} 3b^2 &= a^2 \\ 3b^2 &= a \cdot a \quad | \div a \\ \tfrac{3b^2}{a} &= a \\ 3 \cdot \left(\tfrac{b^2}{a}\right)^2 &= a \end{aligned}$$

$$\Rightarrow 3 \mid a$$

We denote (resp. equate) the coefficient of the 3 with k

$$3 \cdot \underbrace{\left(\frac{b^2}{a}\right)}_{:= k} = a$$

Thus $3k = a$.

[14] Page 99, exercises 212, a) till d)

5. Isolating the denominator variable: We now insert our result for a in (1).

$$\begin{aligned} 3b^2 &= a^2 \\ 3b^2 &= (3k)^2 \\ 3b^2 &= 9k^2 \quad \mid \div 3 \\ b^2 &= 3k^2 \end{aligned}$$

$\Rightarrow 3 \mid b^2$, since the 3 is a prime factor of b^2:

$$b^2 = \underbrace{3}_{\text{divisor}} \cdot k$$

6. From $3 \mid b^2 \Rightarrow 3 \mid b$:

$$\begin{aligned} b^2 &= 3k^2 \quad \mid \div b \\ b &= 3 \cdot \frac{k^2}{b} \end{aligned}$$

$$\Rightarrow 3 \mid b \quad ↯$$

$3 \mid a$ and $3 \mid b$ is a contradiction to the assumption, that a and b are relatively prime.

$\Rightarrow \sqrt{3} \notin \mathbb{Q}$.

□

2. Proove: $\sqrt{6} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt{6} \in \mathbb{Q}$.

2. Then follows ($\sqrt{6} = \sqrt{2 \cdot 3} = \sqrt{2} \cdot \sqrt{3}$), a and b are reduced and coprime:

$$\begin{aligned} \sqrt{2} \cdot \sqrt{3} &= \frac{a}{b} \\ \sqrt{2} \cdot \sqrt{3} &= \frac{a}{b} \quad \mid (\ )^2 \\ (\sqrt{2} \cdot \sqrt{3})^2 &= \left(\frac{a}{b}\right)^2 \end{aligned}$$

$$\begin{aligned} \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{3} \cdot \sqrt{3} &= \frac{a^2}{b^2} \\ 2 \cdot 3 &= \frac{a^2}{b^2} \quad \mid \cdot b^2 \\ 2 \cdot 3 \cdot b^2 &= a^2 \end{aligned}$$

The 2 is a divisor of a^2: $2 \mid a^2$.

3. Isolating the numerator variable:

$$2 \cdot 3 \cdot b^2 = a^2 \quad \mid \div a$$

$$\frac{2 \cdot 3 \cdot b^2}{a} = a$$
$$\frac{2}{1} \cdot \frac{3 \cdot b^2}{a} = a$$
$$\underbrace{2}_{\text{divisor of } a} \cdot \frac{3b^2}{a} = a$$

$\Rightarrow 2 \mid a$, the numerator is even.

4. Isolating the denominator variable: a is also an even number, i.e. $a = 2k$.

$$\begin{aligned} 6b^2 &= a^2 \\ 6b^2 &= (2k)^2 \\ 6b^2 &= 4k^2 \qquad \mid \div 2 \\ 3b^2 &= 2k^2 \end{aligned}$$

From $3 \underbrace{\cdot \, b^2 =}_{\text{even part}} 2k^2$ follows $2 \mid b^2$.

5.

$$\begin{aligned} 3b^2 &= 2k^2 \quad \mid \div b \\ \frac{3b\not{2}}{\not{b}} &= \frac{2k^2}{b} \quad \mid \div 3 \\ b &= \frac{2k^2}{b \cdot 3} \\ b &= \frac{2}{1} \cdot \frac{k^2}{3b} \\ b &= 2 \cdot \frac{k^2}{3b} \\ &\Rightarrow 2 \mid b \ \lightning \end{aligned}$$

If $2 \mid a$ and $2 \mid b$ apply, then the numerator and denominator have a commmon divisor. That contradicts the assumption, that a and b are relatively prime.

q. e. d.

3. Show: $\sqrt[3]{5} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt[3]{5}$ is rational, a and b are shortened and relatively prime. We reformulate the equation and then isolate the numerator variable and the denominator variable.

$$\begin{aligned} \sqrt[3]{5} &= \frac{a}{b} \quad \mid (\ )^3 \\ 5 &= \frac{a^3}{b^3} \quad \mid \cdot b^3 \\ 5b^3 &= a^3 \end{aligned} \tag{1}$$

2. Isolating the numerator variable:

$$5 \cdot b^3 = a^3$$

$$\Rightarrow 5 \text{ is prime divisor of } a^3 : 5 \mid a^3$$

$$\begin{aligned} 5 \cdot b^3 &= a^3 \quad \mid \div a^2 \\ 5 \cdot \frac{b^3}{a^2} &= a \\ &\Rightarrow 5 \mid a \end{aligned}$$

3. Isolating the denominator variable: From (1) we know, that 5 is a divisor of

$a^3 (5 \mid a^3)$, therefore we set $a^3 = 5k$.

$$5b^3 = a^3$$
$$5b^3 = (5k)^3$$
$$5b^3 = 5^3 \cdot k^3$$
$$5b^3 = 125k^3 \quad | \div 5$$
$$b^3 = 25 \cdot k^3$$
$$b^3 = 5 \cdot 5 \cdot k^3$$
$$\Rightarrow 5 \mid b^3$$

4.

$$b^3 = 25k^3 \quad | \div b^2$$
$$\frac{b^{\not{3}1}}{\not{b^2}} = 25 \cdot \frac{k^3}{b^2}$$
$$b = 5 \cdot 5 \cdot \frac{k^3}{b^2}$$
$$\Rightarrow 5 \mid b \ ↯$$

As $5 \mid a$ and $5 \mid b$ applies, a and b are not relatively prime.

■

4. Show: $\sqrt[3]{6}$ is irrational.

Proof: indirect.

Assumption: $\sqrt[3]{6}$ is rational. Then applies, with a and b shortened and coprime present: $\sqrt[3]{6} = \frac{a}{b}$, $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.

1. Transforming and isolating of the numerator variable:

$$\sqrt[3]{6} = \frac{a}{b}$$
$$\sqrt[3]{6} = \frac{a}{b} \quad | \ (\ )^3$$
$$6 = \left(\frac{a}{b}\right)^3$$
$$6 = \frac{a^3}{b^3} \quad | \cdot b^3$$
$$6b^3 = a^3 \tag{1}$$
$$(2 \cdot 3) \cdot b^3 = a^3 \quad | \ ^3$$
$$\frac{a^{\not{3}}}{\not{a^2}} = (2 \cdot 3) \cdot \frac{b^3}{a^2}$$
$$a = 3 \cdot \underbrace{\left(2 \cdot \frac{b^3}{a^2}\right)}_{:= k} \tag{2}$$

$\Rightarrow 3 \mid a$, as the 3 is a prime divisor of a.

2. Isolating the numerator variable b: From (2) follows $a = 3k$. Inserting into (1):

$$6b^3 = a^3$$
$$6b^3 = (3k)^3$$
$$6b^3 = 27k^3 \quad | \div 3$$
$$2b^3 = 9k^3 \quad | \div 2$$
$$b^3 = \frac{9}{2} \cdot k^3 \quad | \div b^2$$
$$b = \frac{9}{2} \cdot \frac{k^3}{b^2}$$

$$b = \frac{3}{1} \cdot \frac{3}{1} \cdot \frac{1}{2} \cdot \frac{k^3}{b^2}$$
$$b = 3 \cdot \left(\frac{3}{2} \cdot \frac{k^3}{b^2}\right)$$

$$\Rightarrow 3 \mid b \ ↯$$

Since numerator and denominator, a and b are not relatively prime as assumed, and $\sqrt[3]{6}$ cannot be described as a fraction, the $\sqrt[3]{6}$ must be, according to the law of the excluded middle, irrational: $\sqrt[3]{6} \in \mathbb{I}$.

$\sqrt[3]{6} \notin \mathbb{Q}$ resp. $\sqrt[3]{6} \in \mathbb{I}$ resp. $\sqrt[3]{6} \in \mathbb{R}$.

□

5. Show: $\sqrt{7} \in \mathbb{I}$.

Proof: indirect.
1. Assumption: $\sqrt{7} \in \mathbb{Q}. \Rightarrow \sqrt{7} = \frac{a}{b}$.
2. W.l.o.g.: $a \perp b$.
3. $\sqrt{7} = \frac{a}{b} \overset{()^2}{\Leftrightarrow} 7 = \frac{a^2}{b^2} \overset{\cdot b^2}{\Leftrightarrow} 7b^2 = a^2 \overset{\div a}{\Leftrightarrow} 7 \cdot \frac{b^2}{a} = a \overset{k:=\frac{b^2}{a}}{\Leftrightarrow} 7 \cdot k = a \Rightarrow 7 \mid a$.
4. $7 \cdot b^2 = a^2 \overset{a:=7\cdot k}{\Longleftrightarrow} 7b^2 = (7k)^2 \Leftrightarrow 7b^2 = 49k^2 \overset{\div 7}{\Leftrightarrow} b^2 = 7k^2 \overset{\div b}{\Leftrightarrow} b = 7 \cdot \frac{k^2}{b} \Rightarrow 7 \mid b$.
5. $\Rightarrow a \not\perp b \Rightarrow$ ↯ to 2.

□

6. Show: $\sqrt{11} \notin \mathbb{Q}$.

Proof: indirect.

1. Assumption: $\sqrt{11} \in \mathbb{Q}. \Rightarrow \sqrt{11} = \frac{a}{b}$.
2. W.l.o.g.: $a \perp b$.
3. $\sqrt{11} = \frac{a}{b} \overset{()^2}{\Leftrightarrow} 11 = \frac{a^2}{b^2} \overset{\cdot b^2}{\Leftrightarrow} 11b^2 = a^2 \overset{\div a}{\Leftrightarrow} 11 \cdot \frac{b^2}{a} = a \overset{k:=\frac{b^2}{a}}{\Leftrightarrow} 11 \cdot k = a \Rightarrow 11 \mid a$.
4. $11 \cdot b^2 = a^2 \overset{a:=11\cdot k}{\Longleftrightarrow} 11b^2 = (11k)^2 \Leftrightarrow 11b^2 = 121k^2 \overset{\div 11}{\Leftrightarrow} b^2 = 11k^2 \overset{\div b}{\Leftrightarrow} b = 11 \cdot \frac{k^2}{b}$
   $\Rightarrow 11 \mid b$.
5. $\Rightarrow a \not\perp b \Rightarrow$ ↯ to 2.

□

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